REPORT PO 52-4601

MICROWAVE OPTICS III

Theoretical Studies on the Aberrations of a Microwave Reflector

FOCAL-PLANE PHASE DISTRIBUTION AFTER OFF-AXIS REFLECTION BY A PARABOLOID, including the Application of a Technique for the Reversion of Double Series

February 1952

PARKE MATHEMATICAL LABORATORIES, INC.
CONCORD, MASSACHUSETTS

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MICROWAVE OPTICS III: "FOCAL-PLANE PHASE DISTRIBUTION AFTER OFF-AXIS REFLECTION BY A PARABOLOID, including the application of a technique for the reversion of double series."

This report is divided into two sections. In the first section, we determine the normal rectilinear congruence generated by rays emanating from a point source near the focus of and reflected by a paraboloid of revolution. In the second and major section we obtain the relative phase of the radiation on the focal plane and calculate the power series expansion of this "Relative Phase Function" to the third order in focal-plane-Cartesian coordinates.

INTRODUCTION

The following report covers a theoretical phase of the Antenna Laboratories' program of study of methods of producing a $\csc^2\theta$ pattern with the FPS-3 paraboloid antenna. The related Antenna Laboratory Memoranda are:

1. Spencer, Roy C.: "Phase Errors of Reflector with Point Source Feed," 18 September 1951.
2. Sletten, C. J.: "Some Proofs and Computations for Designing $\csc^2\theta$ Patterns with a Cut Paraboloid," (including unpublished notes by F. S. Holt.) November 1951 - Pencil Memo.
3. Hiatt, R. and F. Holt, "Conference on AN/FPS-3 Antenna" 16 November 1951.
4. Spencer, R. C.: "Summary of Conferences on Microwave Reflector Design (29 Nov. 1951)" - 30 November 1951.
5. Sletten, C. J.: "Pattern Distortion in Parabolas Caused by Moving the Feed Away from the Focus," 4 December 1951.

INTRODUCTION

6. Hiatt, Ralph E.: "AN/MPS-7," 17 December 1951.

7. Hiatt, Ralph E.: "AN/MPS-7," 2 January 1952.

8. Sletten, C. J.: "Further Analysis of Feed Problem for FPS-3 Antenna," 2 January 1952.

9. Sletten, C. J.: "Further Design Suggestions for $Csc^2\theta$ FPS-3," 24 January 1952.

The far field pattern of antenna is given by the expression,

$$g(u,v) = \int a(x,y)e^{i\phi(x,y)} e^{i(vx+vy)} dxdy$$

where integration is carried out over an aperture and where,

$a(x,y)$.=. amplitude distribution
$\phi(x,y)$.=. phase distribution
$g(u,v)$.=. electric field intensities in direction (u,v)
u,v .=. direction cosines

Departure of $\phi(x,y)$ from a constant is known as phase error. C. J. Sletten, in memorandum No. 9, cited above, developed an approximate series expression for the phase distribution $\phi(x,y)$ in the aperture plane. In a joint conference between the Antenna Laboratory and the Parke Mathematical Laboratories, he emphasized the desirability of carrying out the series approximation more accurately. This suggestion met with the approval of Dr. Spencer as providing a problem of general importance to the research program of the laboratory as well as being of immediate importance in connection with the FPS-3 study.

This report may be considered another in a series of reports on geometrical aspects of microwave optics.[1] The first section of the report

1) Two of the previous reports which are fundamental are:
N.G.Parke: "Microwave Optics I: Rectilinear Congruences," Report PO 50-3404, November 5, 1949.
N.G. Parke: "Microwave Optics II: Electromagnetic Aspects of the Focal Region," Report PO-5404, February 20, 1950.
Other related reports have been prepared under contract AF 19(122)-484.

INTRODUCTION

develops an expression for the distance along any ray from the point source to the focal plane as well as an expression for the rectangular coordinates of the intersection of this ray with the focal plane.

The mathematical difficulty arises from the fact that expressions for the phase and for the coordinates of intersection are parametric. The parameters are polar coordinates to points on the paraboloid. What we desire is the expression for the phase as a function of the coordinates of intersection. The three parametric expressions are too involved for a satisfactory elimination of the "paraboloidal-coordinates." The second part of the report treats our method of working around this difficulty to obtain a double series expression for this phase function in terms of Cartesian coordinates in this focal plane. The method is of general applicability and interesting in itself. We carried out the series up to and including third order terms.

THE REFLECTING SURFACE (REFERENCE SURFACE)

A. THE DEFINING EQUATIONS OF THE CONGRUENCE OF REFLECTED RAYS

1. Parametric equations of the reflecting surface.

The reflecting surface is a paraboloid of revolution with the x^1-axis as the axis of symmetry. The radius of the paraboloid at the (x^2,x^3)-plane will be $\underline{2a}$, where $\underline{a}$ is the distance from the vertex to the focus of the paraboloid. Further we will take $x^1 = 0$ as the focal point; then the parametric equations are given by

$$\begin{aligned} x^1 &= \frac{1}{4a}(\rho^2 - 4a^2) \\ x^2 &= \rho\cos\theta \qquad ; \\ x^3 &= \rho\sin\theta \end{aligned} \tag{1}$$

where (x^1,ρ,θ) are the cylindrical coordinates of a point on the paraboloid.

Fig.1. Parametric representation of the Paraboloid

The equation corresponding to parametric equations (1) in terms of x^1,x^2,x^3, for the paraboloid is

$$x^1 = \frac{1}{4a}[(x^2)^2 + (x^3)^2 - 4a^2] \tag{1a}$$

2. The unit vector $\underline{\vec{r}}'$ in the direction of propagation of the incoming wave.

The incoming rays are issuing from a point source located in the (x^1,x^3)-plane at (e,d), say. Evidently we can obtain the vector from (e,d) to the paraboloid as the difference between the vector from the origin (also the focus) to the paraboloid and the vector from the origin to the point (e,d), that is (see Figure 2)

THE REFLECTING SURFACE (REFERENCE SURFACE)

$$\vec{r}_3 = \vec{r}_1 - \vec{r}_2 \,. \tag{2}$$

Since

$$\vec{r}\,' = \frac{\vec{r}_3}{|\vec{r}_3|} \,, \tag{3}$$

we have a simple way to obtain $\vec{r}\,'$.
So that with

$$\begin{aligned} \vec{r}_1 &= (x^1, \rho\cos\theta, \rho\sin\theta) \\ \vec{r}_2 &= (e, 0, d), \end{aligned} \tag{4}$$

Fig.2. Unit vector in the direction of the incoming wave.

we have from (2), (3) and (4)

$$\vec{r}\,' = \frac{(x^1 - e, \rho\cos\theta, \rho\sin\theta - d)}{\sqrt{(x^1-e)^2 + \rho^2\cos^2\theta + \rho^2\sin^2\theta - 2\rho\sin\theta d + d^2}} \;; \tag{5}$$

or in terms of ρ and θ alone

$$\vec{r}\,' = \frac{\left(\frac{\rho^2 - 4a^2}{4a} - e, \rho\cos\theta, \rho\sin\theta - d\right)}{\sqrt{\left(\frac{\rho^2-4a^2}{4a} - e\right)^2 + \rho^2 - 2\rho\sin\theta d + d^2}} \;; \tag{6}$$

or in terms of the Cartesian coordinates x^1, x^2, x^3

$$\vec{r}\,' = \frac{(x^1 - e, x^2, x^3 - d)}{\sqrt{(x^1-e)^2 + (x^2)^2 + (x^3-d)^2}} \,. \tag{6'}$$

In (6') x^1, x^2, x^3 must satisfy (1a).

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3. The expression for $\vec{n}$, the unit vector normal to the (inside) reflecting side of the paraboloid.

Evidently a vector in the correct direction would be (see Figure 3)

$[\vec{x}_\rho \vec{x}_\theta]$; where $[\vec{r}_1 \vec{r}_2]$.=. the vector product of $\vec{r}_1$ and $\vec{r}_2$ in the Gans notation. And

$$\vec{x}_\rho \equiv \frac{\partial \vec{x}}{\partial \rho};\quad \vec{x}_\theta \equiv \frac{\partial \vec{x}}{\partial \theta};$$

$$\vec{x} \;.=.\; x^1 i_1 + x^2 i_2 + x^3 i_3 \;.=.\; (x^1, x^2, x^3).$$

Fig.3. __The unit vector to the inside of the paraboloid.__

So that

$$\vec{n} = \frac{[\vec{x}_\rho \vec{x}_\theta]}{|[\vec{x}_\rho \vec{x}_\theta]|} \tag{7}$$

must be the unit vector required. Then since

$$\begin{aligned} \vec{x}_\rho &= \left(\tfrac{\rho}{2a}, \cos\theta, \sin\theta\right) \\ \vec{x}_\theta &= (0, -\rho\sin\theta, \rho\cos\theta) \end{aligned}$$

$\vec{n}$ must be given by

$$\vec{n} = \frac{\frac{\rho}{2a}\begin{vmatrix} 2a i_1 & i_2 & i_3 \\ \rho & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{vmatrix}}{|[\vec{x}_\rho \vec{x}_\theta]|}$$

$$\vec{n} = \frac{1}{\sqrt{4a^2+\rho^2}}(2a, -\rho\cos\theta, -\rho\sin\theta) \tag{8}$$

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4. The unit vector $\vec{r}''$ in the direction of propagation of the reflected wave.

Using the standard vector expression for the law of reflection, viz.

$$\vec{r}'' = \vec{r}' - 2(\vec{n}\cdot\vec{r}')\vec{n} \,, \tag{9}$$

we have, with equations (6) and (8),

$$\vec{r}'' = \frac{\left(\frac{\rho^2-4a^2}{4a} - e, \rho\cos\theta, \rho\sin\theta - d\right)}{\sqrt{\left(\frac{\rho^2-4a^2}{4a} - e\right)^2 + \rho^2 - 2\rho d\sin\theta + d^2}} - \frac{2\left(\frac{\rho^2-4a^2}{2} - 2ae - \rho^2 + \rho d\sin\theta\right)(2a, -\rho\cos\theta, -\rho\sin\theta)}{(4a^2+\rho^2)\sqrt{\left(\frac{\rho^2-4a^2}{4a} - e\right)^2 + \rho^2 - 2\rho d\sin\theta + d^2}} \,; \tag{10}$$

or, performing the operations and simplifying,

$$\vec{r}'' = \frac{\left[\frac{(\rho^2+4a^2)^2 - 4ae(\rho^2-4a^2) - 16a^2 d\rho\sin\theta}{4a}, \rho\cos\theta(2\rho d\sin\theta - 4ae), \rho\sin\theta(2\rho d\sin\theta - 4ae) - d(\rho^2+4a^2)\right]}{(4a^2+\rho^2)\sqrt{\left(\frac{\rho^2-4a^2}{4a} - e\right)^2 + \rho^2 - 2\rho d\sin\theta + d^2}} \,. \tag{10'}$$

With

$\vec{r}'' = (X^1(\rho,\theta),\ X^2(\rho,\theta),\ X^3(\rho,\theta)\ )$,
where the X^i are direction cosines of the unit vector $\vec{r}''$ in the direction of propagation of the reflected wave,

THE REFLECTING SURFACE (REFERENCE SURFACE)

we have the direction cosines of the lines making up the congruence

$$\begin{cases} X^1(\rho,\theta) = \dfrac{(\rho^2+4a^2)^2 - 4ae(\rho^2-4a^2) - 16a^2\rho d\sin\theta}{4a(4a^2+\rho^2)S^{1/2}} \\ X^2(\rho,\theta) = \dfrac{\rho\cos\theta(2d\rho\sin\theta - 4ae)}{(4a^2+\rho^2)S^{1/2}} \\ X^3(\rho,\theta) = \dfrac{\rho\sin\theta(2d\rho\sin\theta - 4ae) - d^2(\rho^2+4a^2)}{(4a^2+\rho^2)S^{1/2}} \end{cases} \quad , \qquad (11)$$

$$\text{where } S \;.=.\; \left(\frac{\rho^2+4a^2}{4a} - e\right)^2 + \rho^2 - 2\rho d\sin\theta + d^2 \quad ,$$

and with the reference surface

$$\begin{aligned} x^1(\rho,\theta) &= \frac{\rho^2-4a^2}{4a} \\ x^2(\rho,\theta) &= \rho\cos\theta \\ x^3(\rho,\theta) &= \rho\sin\theta \end{aligned} \qquad (1)$$

we have defined the normal rectilinear congruence generated by the reflected rays. This may now be used to determine the caustic (focal) surface and other properties of the reflection.

II. THE FOCAL PLANE

A. THE PHASE DISTRIBUTION (RELATIVE PHASE FUNCTION) ON THE FOCAL PLANE

The task which we will require of the congruence defined by the equations (11) and (1) will be to determine the trace of the rays on the focal plane and the expression for the relative phase on the focal plane - the relative phase function.

1. The trace of the intersection of rays with the focal plane.

The first step in obtaining the trace on the focal plane of the reflected rays is to determine t the distance along a reflected ray from the paraboloid to the focal plane. If we wish to find the coordinates y^1, y^2, y^3 of points a distance t' along a reflected ray we obtain them as

$$y^i = x^i + t'X^i, \quad (i = 1,2,3) \tag{12}$$

where x^i are the rectangular coordinates of a point on the paraboloid from which we start and X^i are the direction numbers of the reflected ray along which we measure. Thus the value $t' = t$ for which we will arrive at the focal plane should be determined by setting $y^1 = 0$; that is

$$t = -\frac{x^1}{X^1} = -\frac{\rho^2-4a^2}{4a} \cdot \frac{4a(4a^2+\rho^2)S^{1/2}}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta}, \tag{13}$$

$$\text{where } S = \left(\frac{\rho^2-4a^2}{4a}-e\right)^2+\rho^2-2\rho d\sin\theta+d^2 .$$

The intersection y^i of the rays on the focal plane are then

$$y^i = x^i + tX^i, \quad (i = 2,3)$$

or letting $y^2 = y$, $y^3 = z$, for convenience,

THE FOCAL PLANE

$$y^1 \begin{cases} y = \rho\cos\theta + \rho\cos\theta \dfrac{(\rho^2-4a^2)(4ae-2\rho d\sin\theta)}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta}\,. \\[2ex] z = \rho\sin\theta + \rho\sin\theta \dfrac{(\rho^2-4a^2)(4ae-2\rho d\sin\theta)+\dfrac{d(\rho^4-16a^4)}{\rho\sin\theta}}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta} \end{cases} \qquad (14)$$

or simplified

$$y^1 \begin{cases} y = \rho\cos\theta \dfrac{(\rho^2+4a^2)^2-2\rho d\sin\theta(\rho^2+4a^2)}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta} \\[2ex] z = \rho\sin\theta \dfrac{(\rho^2+4a^2)^2-2\rho d\sin\theta(\rho^2+4a^2)}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta} + \end{cases} \qquad (15)$$

$$\frac{d(\rho^4-16a^4)}{(\rho^2+4a^2)^2-4ae(\rho^2-4a^2)-16a^2\rho d\sin\theta}$$

or in terms of x^i, writing $x_2 = x^2$, $x_3 = x^3$, for convenience,

$$y^1 \begin{cases} y = x_2 \dfrac{(x_2^2+x_3^2+4a^2)(x_2^2+x_3^2+4a^2-2dx_3)}{(x_2^2+x_3^2+4a^2)^2-4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3} \\[2ex] z = x_3 \dfrac{(x_2^2+x_3^2+4a^2)(x_2^2+x_3^2+4a^2-2dx_3)}{(x_2^2+x_3^2+4a^2)^2-4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3} + \end{cases} \qquad (16)$$

$$\frac{d(x_2^4+2x_2^2x_3^2+x_3^4-16a^4)}{(x_2^2+x_3^2+4a^2)^2-4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3}\,.$$

2. The relative phase function on focal plane

The relative phase Λ is given by

$$\Lambda = \psi\frac{c}{2\pi f} = \frac{\psi}{k} \qquad (17)$$

; where c = velocity of light
f = the frequency
ψ = the phase
k = the propagation constant

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and, since Λ is just the distance from the point source to the focal plane along a ray, we have

$$\Lambda = t + |\vec{r}_3| \ . \tag{18}$$

And, since $S^{1/2} = |\vec{r}_3|$,

$$\Lambda = S^{1/2} - \frac{(\rho^4 - 16a^4)S^{1/2}}{(\rho^2+4a^2)^2 - 4ae(\rho^2-4a^2) - 16a^2\rho d \sin\theta} \ . \tag{19}$$

Simplifying this we get

$$\Lambda(\rho,\theta) = S^{1/2}\left[\frac{8a^2(\rho^2+4a^2) - 4ae(\rho^2-4a^2) - 16a^2\rho d\sin\theta}{(\rho^2+4a^2)^2 - 4ae(\rho^2-4a^2) - 16a^2\rho d\sin\theta}\right] \tag{20}$$

or in terms of x_2, x_3

$$\Lambda(x_2,x_3) = \sqrt{(x_1-e)^2 + x_2^2 + (x_3-d)^2} \cdot \left\{\frac{8a^2(x_2^2+x_3^2+4a^2) - 4ae(x_2^2+x_3^2-4a^2) - 16a^2 dx_3}{(x_2^2+x_3^2+4a^2)^2 - 4ae(x_2^2+x_3^2-4a^2) - 16a^2 dx_3}\right\} . \tag{21}$$

THE FOCAL PLANE

B. DERIVATION OF THE POWER SERIES FOR THE RELATIVE PHASE FUNCTION

1. Plan of attack.

The problem of expanding $\Lambda(y,z)$ as a power series in y and z, the Cartesian coordinates on the focal plane, is complicated by the difficulty in inverting (16) to get x_2 and x_3 as functions of y, z. This inversion is theoretically possible but to date we have had no indication of a way to accomplish it.

Another approach which is perfectly sound and possible is this. First expand y and z as a power series in x_2 and x_3

$$\begin{cases} y = a_0+a_2x_2+a_3x_3+a_{22}x_2^2+a_{23}x_2x_3+a_{33}x_3^2+ \dots\dots\dots\dots \\ z = b_0+b_2x_2+b_3x_3+b_{22}x_2^2+b_{23}x_2x_3+b_{33}x_3^2+ \dots\dots\dots\dots \end{cases} \tag{22}$$

Then using a predetermined number of terms revert the series, that is obtain the coefficients of the series for x_2 and x_3 in terms of y and z

$$\begin{cases} x_2 = c_0+c_2y+c_3z+c_{22}y^2+c_{23}yz+c_{33}z^2+ \dots\dots\dots\dots \\ x_3 = f_0+f_2y+f_3z+f_{22}y^2+f_{23}yz+f_{33}z^2+ \dots\dots\dots\dots \end{cases} \tag{23}$$

and with this expansion determine the coefficients of a series in y,z for $\Lambda(y,z)$.

It can be shown that the series for $\Lambda(y,z)$ thus determined will be correct to the degree that (22) and (23) are expanded.

Although this scheme is theoretically sound and actually possible, the work involved is algebraicly tedious. Our actual attack is somewhat different though based on the same principle.

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2. The expansion of Λ, the relative phase function, in a Taylor's series symbolically.

For convenience we adopt the summation convention, viz.

$$\alpha_i x^i \;.=.\; \sum_i \alpha_i x^i \;.=.\; \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3, \quad (i = 1,2,3)$$

and return to the notation $y^2 = y$, $y^3 = z$ and $x^2 = x_2$, $x^3 = x_3$.

Then the power series expansion of $\Lambda(x^i(y^\alpha)) = \Lambda(y^\alpha)$ may be written

$$\Lambda(y^\alpha) = \Gamma + \Gamma_\alpha(y^\alpha - y_0^\alpha) + \Gamma_{\alpha\beta}(y^\alpha - y_0^\alpha)(y^\beta - y_0^\beta) + \Gamma_{\alpha\beta\gamma}(y^\alpha - y_0^\alpha)(y^\beta - y_0^\beta)(y^\gamma - y_0^\gamma) + \cdots \quad (24)$$

$$(\alpha, \beta, \gamma = 2,3)$$

where the subscript zero will denote the value at $y^2 = y_0^2$, $y^3 = y_0^3$, the point in the focal plane about which the expansion is made. In our case we choose the point for which $x^2 = x^3 = 0$, at which Λ is analytic.

Then

$$\Lambda_0 = \Gamma, \quad \frac{\partial\Lambda_0}{\partial y^\alpha} = \Gamma_\alpha, \quad \frac{\partial^2\Lambda_0}{\partial y^\alpha \partial y^\beta} = 2!\,\Gamma_{\alpha\beta}, \quad \frac{\partial^3\Lambda_0}{\partial y^\alpha \partial y^\beta \partial y^\gamma} = 3!\,\Gamma_{\alpha\beta\gamma}, \cdots \quad (25)$$

Thus we have

$$\Lambda(y^\alpha) = \Lambda_0 + \frac{\partial\Lambda_0}{\partial y^\alpha}(y^\alpha - y_0^\alpha) + \frac{1}{2!}\frac{\partial^2\Lambda_0}{\partial y^\alpha \partial y^\beta}(y^\alpha - y_0^\alpha)(y^\beta - y_0^\beta) + \frac{1}{3!}\frac{\partial^3\Lambda_0}{\partial y^\alpha \partial y^\beta \partial y^\gamma} \cdot \quad (26)$$

$$(y^\alpha - y_0^\alpha)(y^\beta - y_0^\beta)(y^\gamma - y_0^\gamma) + \cdots$$

$$(\alpha, \beta, \gamma = 2,3)$$

or without use of the summation convention

$$\begin{aligned}\Lambda(y,z) &= \Lambda_0 + \frac{\partial \Lambda_0}{\partial y}(y-y_0) + \frac{\partial \Lambda_0}{\partial z}(z-z_0) + \frac{1}{2!}\frac{\partial^2 \Lambda_0}{\partial y^2}(y-y_0)^2 \\ &+ \frac{2}{2!}\frac{\partial^2 \Lambda_0}{\partial y \partial z}(y-y_0)(z-z_0) + \frac{1}{2!}\frac{\partial^2 \Lambda_0}{\partial z^2}(z-z_0)^2 + \frac{1}{3!}\frac{\partial^3 \Lambda_0}{\partial y^3}(y-y_0)^3 \qquad (26') \\ &+ \frac{3}{3!}\frac{\partial^3 \Lambda_0}{\partial y^2 \partial z}(y-y_0)^2(z-z_0) + \frac{3}{3!}\frac{\partial^3 \Lambda_0}{\partial y \partial z^2}(y-y_0)(z-z_0)^2 + \frac{1}{3!}\frac{\partial^3 \Lambda_0}{\partial z^3}(z-z_0)^3 + \cdots\end{aligned}$$

but

$$\begin{aligned}&(a) \quad \frac{\partial \Lambda}{\partial y^\alpha} = \frac{\partial \Lambda}{\partial x^i}\frac{\partial x^i}{\partial y^\alpha} \\ &(b) \quad \frac{\partial^2 \Lambda}{\partial y^\alpha \partial y^\beta} = \frac{\partial^2 \Lambda}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta} + \frac{\partial \Lambda}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \qquad (27) \\ &(c) \quad \frac{\partial^3 \Lambda}{\partial y^\alpha \partial y^\beta \partial y^\gamma} = \frac{\partial^3 \Lambda}{\partial x^i \partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma} + \frac{\partial^2 \Lambda}{\partial x^i \partial x^j}\left[\frac{\partial^2 x^i}{\partial y^\alpha \partial y^\gamma}\frac{\partial x^j}{\partial y^\beta} + \frac{\partial x^i}{\partial y^\alpha}\frac{\partial^2 x^j}{\partial y^\beta \partial y^\gamma} \right. \\ &\qquad \left. + \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}\frac{\partial x^j}{\partial y^\gamma}\right] + \frac{\partial \Lambda}{\partial x^i}\frac{\partial^3 x^i}{\partial y^\alpha \partial y^\beta \partial y^\gamma} \qquad (i,j,k,\alpha,\beta,\gamma = 2,3)\end{aligned}$$

which means that we must know the partial derivatives of the x^i with respect to the y^α $(i,\alpha = 2,3)$. These we cannot calculate directly because of the form of the expressions for the y^α, as mentioned above.

3. The partial derivatives of x_2 and x_3 with respect to y and z.

As we saw in the previous paragraph our expansion depends on the calculation of the various partial derivatives of not only Λ but also of the x^i with respect to the y^α $(i,\alpha = 2,3)$. Taking the latter problem first, we have by definition

$$\begin{aligned}&(a) \quad \frac{\partial y^\alpha}{\partial y^\beta} = \delta^\alpha_\beta = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial x^i}{\partial y^\beta}, \quad \text{where } \delta^\alpha_\beta \begin{cases} = 1, \alpha = \beta \\ = 0 \text{ otherwise}\end{cases} \\ &(b) \quad \frac{\partial^2 y^\alpha}{\partial y^\beta \partial y^\gamma} = 0 = \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial y^\beta}\frac{\partial x^j}{\partial y^\gamma} + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma} \qquad (28) \\ &(c) \quad \frac{\partial^3 y^\alpha}{\partial y^\beta \partial y^\gamma \partial y^\delta} = 0 = \frac{\partial^3 y^\alpha}{\partial x^i \partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\beta}\frac{\partial x^j}{\partial y^\gamma}\frac{\partial x^k}{\partial y^\delta} + \frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}\left[\frac{\partial^2 x^i}{\partial y^\beta \partial y^\delta}\frac{\partial x^j}{\partial y^\gamma} + \right. \\ &\qquad \left. \frac{\partial^2 x^i}{\partial y^\beta \partial y^\gamma}\frac{\partial x^j}{\partial y^\delta} + \frac{\partial x^i}{\partial y^\beta}\frac{\partial^2 x^j}{\partial y^\gamma \partial y^\delta}\right] + \frac{\partial y^\alpha}{\partial x^i}\frac{\partial^3 x^i}{\partial y^\beta \partial y^\gamma \partial y^\delta}\end{aligned}$$

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which are equations linear in the highest derivative, and the sole condition on their solution is that the Jacobian does not vanish, which is satisfied here. Thus we have systems of equations which, when solved in order, are linear in the derivative sought. Written out without the subscript notation and summation convention (where $y_2 = \frac{\partial y}{\partial x_2}$, $x_{2y} = \frac{\partial x_2}{\partial y}$, etc.) equations (28) become

$$\text{(a)}\begin{cases} 1 = y_2 x_{2y} + y_3 x_{3y} \\ 0 = z_2 x_{2y} + z_3 x_{3y} \end{cases} \qquad\qquad (29)$$

$$\text{(b)}\begin{cases} 0 = y_2 x_{2z} + y_3 x_{3z} \\ 1 = z_2 x_{2z} + z_3 x_{3z} \end{cases}$$

$$\text{(a)}\begin{cases} 0 = y_{22} x_{2y}^2 + 2y_{23} x_{2y} x_{3y} + y_{33} x_{3y}^2 + y_2 x_{2yy} + y_3 x_{3yy} \\ 0 = z_{22} x_{2y}^2 + 2z_{23} x_{2y} x_{3y} + z_{33} x_{3y}^2 + z_2 x_{2yy} + z_3 x_{3yy} \end{cases}$$

$$\text{(b)}\begin{cases} 0 = y_{22} x_{2y} x_{2z} + y_{23} x_{2y} x_{3z} + y_{23} x_{3y} x_{2z} + y_{33} x_{3y} x_{3z} + y_2 x_{2yz} + y_3 x_{3yz} \\ 0 = z_{22} x_{2y} x_{2z} + z_{23} x_{2y} x_{3z} + z_{23} x_{3y} x_{2z} + z_{33} x_{3y} x_{3z} + z_2 x_{2yz} + z_3 x_{3yz} \end{cases} \qquad (30)$$

$$\text{(c)}\begin{cases} 0 = y_{22} x_{2z}^2 + 2y_{23} x_{2z} x_{3z} + y_{33} x_{3z}^2 + y_2 x_{2zz} + y_3 x_{3zz} \\ 0 = z_{22} x_{2z}^2 + 2z_{23} x_{2z} x_{3z} + z_{33} x_{3z}^2 + z_2 x_{2zz} + z_3 x_{3zz} \end{cases}$$

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$$
\begin{gathered}
\text{(a)}\begin{cases}
0 = y_{222}x_{2y}^3+3y_{223}x_{2y}^2x_{3y}+3y_{332}x_{2y}x_{3y}^2+y_{333}x_{3y}^3+3y_{22}x_{2yy}x_{2y} \\
\quad +3y_{23}(x_{2yy}x_{3y}+x_{2y}x_{3yy})+3y_{33}x_{3yy}x_{3y}+y_2x_{2yyy}+y_3x_{3yyy} \\
0 = z_{222}x_{2y}^3+3z_{223}x_{2y}^2x_{3y}+3z_{332}x_{2y}x_{3y}^2+z_{333}x_{3y}^3+3z_{22}x_{2yy}x_{2y} \\
\quad +3z_{23}(x_{2yy}x_{3y}+x_{2y}x_{3yy})+3z_{33}x_{3yy}x_{3y}+z_2x_{2yyy}+z_3x_{3yyy}
\end{cases} \\
\text{(b)}\begin{cases}
0 = y_{222}x_{2z}^3+3y_{223}x_{2z}^2x_{3z}+3y_{332}x_{3z}^2x_{2z}+y_{333}x_{3z}^3+3y_{22}x_{2zz}x_{2z} \\
\quad +3y_{23}(x_{2zz}x_{3z}+x_{2z}x_{3zz})+3y_{33}x_{3zz}x_{3z}+y_2x_{2zzz}+y_3x_{3zzz} \\
0 = z_{222}x_{2z}^3+3z_{223}x_{2z}^2x_{3z}+3z_{332}x_{3z}^2x_{2z}+z_{333}x_{3z}^3+3z_{22}x_{2zz}x_{2z} \\
\quad +3z_{23}(x_{2zz}x_{3z}+x_{2z}x_{3zz})+3z_{33}x_{3zz}x_{3z}+z_2x_{2zzz}+z_3x_{3zzz}
\end{cases} \\
\text{(c)}\begin{cases}
0 = y_{222}x_{2y}^2x_{2z}+y_{223}(x_{2y}^2x_{3z}+2x_{2y}x_{3y}x_{2z})+y_{332}(x_{3y}^2x_{2z}+2x_{3y}x_{2y}x_{3z}) \\
\quad +y_{333}x_{3y}^2x_{3z}+2[y_{22}x_{2yz}x_{2y}+y_{23}(x_{2yz}x_{3y}+x_{3yz}x_{2y})+y_{33}x_{3yz}x_{3y}] \\
\quad +y_{22}x_{2yy}x_{2z}+y_{23}(x_{2yy}x_{3z}+x_{3yy}x_{2z})+y_{33}x_{3yy}x_{3z}+y_2x_{2yyz}+y_3x_{3yyz} \\
0 = z_{222}x_{2y}^2x_{2z}+z_{223}(x_{2y}^2x_{3z}+2x_{2y}x_{3y}x_{2z})+z_{332}(x_{3y}^2x_{2z}+2x_{3y}x_{3z}x_{2y}) \\
\quad +z_{333}x_{3y}^2x_{3z}+2[z_{22}x_{2yz}x_{2y}+z_{23}(x_{2yz}x_{3y}+x_{3yz}x_{2y})+z_{33}x_{3yz}x_{3y}] \\
\quad +z_{22}x_{2yy}x_{2z}+z_{23}(x_{2yy}x_{3z}+x_{3yy}x_{2z})+z_{33}x_{3yy}x_{3z}+z_2x_{2yyz}+z_3x_{3yyz}
\end{cases} \qquad (31) \\
\text{(d)}\begin{cases}
0 = y_{222}x_{2z}^2x_{2y}+y_{223}(x_{2z}^2x_{3y}+2x_{2z}x_{3z}x_{2y})+y_{332}(x_{3z}^2x_{2y}+2x_{3z}x_{2z}x_{3y}) \\
\quad +y_{333}x_{3z}^2x_{3y}+2[y_{22}x_{2yz}x_{2z}+y_{23}(x_{2yz}x_{3z}+x_{3yz}x_{2z})+y_{33}x_{3yz}x_{3z}] \\
\quad +y_{22}x_{2zz}x_{2y}+y_{23}(x_{2zz}x_{3y}+x_{3zz}x_{2y})+y_{33}x_{3zz}x_{3y}+y_2x_{2zzy}+y_3x_{3zzy} \\
0 = z_{222}x_{2z}^2x_{2y}+z_{223}(x_{2z}^2x_{3y}+2x_{2z}x_{3z}x_{2y})+z_{332}(x_{3z}^2x_{2y}+2x_{3z}x_{2z}x_{3y}) \\
\quad +z_{333}x_{3z}^2x_{3y}+2[z_{22}x_{2yz}x_{2z}+z_{23}(x_{2yz}x_{3z}+x_{3yz}x_{2z})+z_{33}x_{3yz}x_{3z}] \\
\quad +z_{22}x_{2zz}x_{2y}+z_{23}(x_{2zz}x_{3y}+x_{3zz}x_{2y})+z_{33}x_{3zz}x_{3y}+z_2x_{2zzy}+z_3x_{3zzy}
\end{cases}
\end{gathered}
$$

Equations (30) should be 8 equations in 8 unknowns and equations (31) should be 16 equations in 16 unknowns; the symmetry of the cross derivatives eliminates two equations in (30) and 8 equations in (31).

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In order to solve equations (29), (30) and (31) we need expressions for the partials of y and z with respect to x_2 and x_3, which are a direct calculation in the sense that we can differentiate equations (16) directly. This involves the successive derivatives of a quotient and, since there seems to be no table of such a procedure - elementary as it may be - we have included such a table of derivatives up to the fourth order as an appendix to this report[1] in two forms.

---

1) See Appendix A

---

4. The partial derivatives of y and z with respect to x_2 and x_3. Writing equation (16) as

$$\begin{aligned} y &= x_2 Q \\ z &= x_3 Q + R \, , \end{aligned} \qquad (32)$$

where

$$Q \;.{=}.\; \left[\frac{(x_2^2+x_3^2+4a^2)(x_2^2+x_3^2+4a^2-2dx_3)}{(x_2^2+x_3^2+4a^2)^2-4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3} \right]$$

$$R \;.{=}.\; \left[\frac{d(x_2^4+2x_2^2x_3^2+x_3^4-16a^4)}{(x_2^2+x_3^2+4a^2)^2-4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3} \right] ,$$

we have for the partial derivatives of y and z with respect to x_2 and x_3 the following expressions:

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$$
\begin{aligned}
y_2 &= Q + x_2\frac{\partial Q}{\partial x_2} \\
y_3 &= x_2\frac{\partial Q}{\partial x_3} \\
y_{22} &= 2\frac{\partial Q}{\partial x_2} + x_2\frac{\partial^2 Q}{\partial x_2^2} \\
y_{23} &= \frac{\partial Q}{\partial x_3} + x_2\frac{\partial^2 Q}{\partial x_2 \partial x_3} \\
y_{33} &= x_2\frac{\partial^2 Q}{\partial x_3^2} \\
y_{222} &= 3\frac{\partial^2 Q}{\partial x_2^2} + x_2\frac{\partial^3 Q}{\partial x_2^3} \\
y_{223} &= 2\frac{\partial^2 Q}{\partial x_2 \partial x_3} + x_2\frac{\partial^3 Q}{\partial x_2^2 \partial x_3} \\
y_{332} &= \frac{\partial^2 Q}{\partial x_3^2} + x_2\frac{\partial^3 Q}{\partial x_2 \partial x_3^2} \\
y_{333} &= x_2\frac{\partial^3 Q}{\partial x_3^3}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
z_2 &= x_3\frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_2} \\
z_3 &= Q + x_3\frac{\partial Q}{\partial x_3} + \frac{\partial R}{\partial x_3} \\
z_{22} &= x_3\frac{\partial^2 Q}{\partial x_2^2} + \frac{\partial^2 R}{\partial x_2^2} \\
z_{23} &= \frac{\partial Q}{\partial x_2} + x_3\frac{\partial^2 Q}{\partial x_2 \partial x_3} + \frac{\partial^2 R}{\partial x_2 \partial x_3} \\
z_{33} &= 2\frac{\partial Q}{\partial x_3} + x_3\frac{\partial^2 Q}{\partial x_3^2} + \frac{\partial^2 R}{\partial x_3^2} \\
z_{222} &= x_3\frac{\partial^3 Q}{\partial x_2^3} + \frac{\partial^3 R}{\partial x_2^3} \\
z_{223} &= \frac{\partial^2 Q}{\partial x_2^2} + x_3\frac{\partial^3 Q}{\partial x_2^2 \partial x_3} + \frac{\partial^3 R}{\partial x_2^2 \partial x_3} \\
z_{332} &= 2\frac{\partial^2 Q}{\partial x_3 \partial x_2} + x_3\frac{\partial^3 Q}{\partial x_2 \partial x_3^2} + \frac{\partial^3 R}{\partial x_2 \partial x_3^2} \\
z_{333} &= 3\frac{\partial^2 Q}{\partial x_3^2} + x_3\frac{\partial^3 Q}{\partial x_3^3} + \frac{\partial^3 R}{\partial x_3^3}
\end{aligned}
\tag{34}
$$

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In order to evaluate these partial derivatives of y and z at (y_0,z_c), we need the evaluation of the partial derivatives of Q and R at (y_0,z_0). Letting

$$Q = \frac{N}{D}$$

$$R = \frac{N'}{D}$$

we first have, for the values of N, N' and D and their partial derivatives with respect to x_2 and x_3 at (y_0,z_0), the following expressions:

$$\begin{array}{ll}
N \,\big|_{0,0} = (x_2^2+x_3^2+4a^2)(x_2^2+x_3^2+4a^2-2dx_3)\,\big|_{0,0} = 16a^4 & \\
N_2 \,\big|_{0,0} = 2x_2(x_2^2+x_3^2+4a^2-2dx_3)+2x_2(x_2^2+x_3^2+4a^2)\,\big|_{0,0} = 0 & \\
N_3 \,\big|_{0,0} = 2x_3(x_2^2+x_3^2+4a^2-2dx_3)+(2x_3-2d)(x_2^2+x_3^2+4a^2)\,\big|_{0,0} = -8a^2d & \\
N_{22} \,\big|_{0,0} = 12x_2^2+4x_3^2+16a^2-4dx_3\,\big|_{0,0} = 16a^2 & \\
N_{23} \,\big|_{0,0} = 8x_2x_3-4dx_2\,\big|_{0,0} = 0 & (35) \\
N_{33} \,\big|_{0,0} = 12x_3^2+4x_2^2-12dx_3+16a^2\,\big|_{0,0} = 16a^2 & \\
N_{222} \,\big|_{0,0} = 24x_2\,\big|_{0,0} = 0 & \\
N_{223} \,\big|_{0,0} = 8x_3-4d\,\big|_{0,0} = -4d & \\
N_{332} \,\big|_{0,0} = +8x_2\,\big|_{0,0} = 0 & \\
N_{333} \,\big|_{0,0} = 24x_3-12d\,\big|_{0,0} = -12d &
\end{array}$$

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$$
\begin{aligned}
N' \quad &\Big|_{0,0} = d(x_2^4+2x_2^2x_3^2+x_3^4-16a^4 \Big|_{0,0} = -16a^4 d \\
N_2' \quad &\Big|_{0,0} = 4dx_2^3+4dx_2x_3^2 \Big|_{0,0} = 0 \\
N_3' \quad &\Big|_{0,0} = 4dx_3^3+4dx_3x_2^2 \Big|_{0,0} = 0 \\
N_{22}' \quad &\Big|_{0,0} = 12dx_2^2+4dx_3^2 \Big|_{0,0} = 0 \\
N_{23}' \quad &\Big|_{0,0} = 8dx_2x_3 \Big|_{0,0} = 0 \\
N_{33}' \quad &\Big|_{0,0} = 12dx_3^2+4dx_2^2 \Big|_{0,0} = 0 \\
N_{222}' \quad &\Big|_{0,0} = 24dx_2 \Big|_{0,0} = 0 \\
N_{223}' \quad &\Big|_{0,0} = 8dx_3 \Big|_{0,0} = 0 \\
N_{332}' \quad &\Big|_{0,0} = 8dx_2 \Big|_{0,0} = 0 \\
N_{333}' \quad &\Big|_{0,0} = 24dx_3 \Big|_{0,0} = 0
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
D \quad &\Big|_{0,0} = (x_2^2+x_3^2+4a^2)^2 -4ae(x_2^2+x_3^2-4a^2)-16a^2dx_3 \Big|_{0,0} = 16a^3(a+e) \\
D_2 \quad &\Big|_{0,0} = 4x_2^3+4x_2x_3^2+16x_2a^2-8aex_2 \Big|_{0,0} = 0 \\
D_3 \quad &\Big|_{0,0} = 4x_3^3+4x_2^2x_3+16x_3a^2-8aex_3-16a^2d \Big|_{0,0} = -16a^2d \\
D_{22} \quad &\Big|_{0,0} = 12x_2^2+4x_3^2+16a^2-8ae \Big|_{0,0} = 8a(2a-e) \\
D_{23} \quad &\Big|_{0,0} = 8x_2x_3 \Big|_{0,0} = 0 \\
D_{33} \quad &\Big|_{0,0} = 12x_3^2+4x_2^2+16a^2-8ae \Big|_{0,0} = 8a(2a-e) \\
D_{222} \quad &\Big|_{0,0} = 24x_2 \Big|_{0,0} = 0 \\
D_{223} \quad &\Big|_{0,0} = 8x_3 \Big|_{0,0} = 0 \\
D_{332} \quad &\Big|_{0,0} = 8x_2 \Big|_{0,0} = 0 \\
D_{333} \quad &\Big|_{0,0} = 24x_3 \Big|_{0,0} = 0
\end{aligned}
\tag{37}
$$

Using these expressions in the table of derivatives of a quotient found in Appendix A we obtain, somewhat laboriously, that

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$$Q_0 = \frac{a}{a+e} \qquad\qquad R_0 = -d\frac{a}{(a+e)}$$

$$\frac{\partial Q_0}{\partial x_2} = 0 \qquad\qquad \frac{\partial R_0}{\partial x_2} = 0$$

$$\frac{\partial Q_0}{\partial x_3} = d\frac{(a-e)}{2a(a+e)^2} \qquad\qquad \frac{\partial R_0}{\partial x_3} = -\frac{d^2}{(a+e)^2}$$

$$\frac{\partial^2 Q_0}{\partial x_2^2} = \frac{3e}{2a(a+e)^2} \qquad\qquad \frac{\partial^2 R_0}{\partial x_2^2} = d\frac{(2a-e)}{2a(a+e)^2} \tag{38}$$

$$\frac{\partial^2 Q_0}{\partial x_2 \partial x_3} = 0 \qquad\qquad \frac{\partial^2 R_0}{\partial x_2 \partial x_3} = 0$$

$$\frac{\partial^2 Q_0}{\partial x_3^2} = \frac{3ae(a+e)+2d^2(a-e)}{2a^2(a+e)^3} \qquad\qquad \frac{\partial^2 R_0}{\partial x_3^2} = \frac{d(2a-e)(a+e)-4d^3}{2a(a+e)^3}$$

$$\frac{\partial^3 R_0}{\partial x_2^3} = 0$$

$$\frac{\partial^3 R_0}{\partial x_2^2 \partial x_3} = \frac{d^2(2a-e)}{a^2(a+e)^3}$$

$$\frac{\partial^3 R_0}{\partial x_2 \partial x_3^2} = 0$$

$$\frac{\partial^3 R_0}{\partial x_3^3} = \frac{3d^2(2a-e)(a+e)-6d^4}{a^2(a+e)^4}$$

With the above values for Q and R and their partial derivatives at $x_2 = x_3 = 0$, we can now obtain the value of y and z and their partial derivatives with respect to x_2 and x_3 at $x_2 = x_3 = 0$, which are

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$$\begin{cases}
y_{o} = 0 \\[2ex]
y_{o_2} = \dfrac{a}{a+e} \\[2ex]
y_{o_3} = 0 \\[2ex]
y_{o_{22}} = 0 \\[2ex]
y_{o_{23}} = d\dfrac{(a-e)}{2a(a+e)^2} \\[2ex]
y_{o_{33}} = 0 \\[2ex]
y_{o_{222}} = \dfrac{9e}{2a(a+e)^2} \\[2ex]
y_{o_{223}} = 0 \\[2ex]
y_{o_{332}} = \dfrac{3ae(a+e)+2d^2(a-e)}{2a^2(a+e)^3} = \dfrac{3e}{2a(a+e)^2} + d^2\dfrac{(a-e)}{a^2(a+e)^3} \\[2ex]
y_{o_{333}} = 0\ ,
\end{cases} \qquad (39)$$

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$$
\left\{
\begin{aligned}
z_o &= -d\frac{a}{a+e} \\
z_{o_2} &= 0 \\
z_{o_3} &= -\frac{d^2-a^2-ae}{(a+e)^2} = \frac{a}{(a+e)} - \frac{d^2}{(a+e)^2} \\
z_{o_{22}} &= d\frac{(2a-e)}{2a(a+e)^2} \\
z_{o_{23}} &= 0 \\
z_{o_{33}} &= \frac{(4a-3e)(a+e)d-4d^3}{2a(a+e)^3} = d\frac{(4a-3e)}{2a(a+e)^2} - \frac{2d^3}{a(a+e)^3} \\
z_{o_{222}} &= 0 \\
z_{o_{223}} &= \frac{3ae(a+e)+2d^2(2a-e)}{2a^2(a+e)^3} = \frac{3e}{2a(a+e)^2} + d^2\frac{(2a-e)}{a^2(a+e)^3} \\
z_{o_{332}} &= 0 \\
z_{o_{333}} &= \frac{9ae(a+e)^2+6d^2(3a-2e)(a+e)-12d^4}{2a^2(a+e)^4} = \\
&\quad \frac{9e}{2a(a+e)^2} + \frac{3d^2(3a-2e)}{a^2(a+e)^3} - \frac{6d^4}{a^2(a+e)^4} .
\end{aligned}
\right.
\tag{40}
$$

5. The value of the partial derivatives of x_2 and x_3 with respect to y and z at (y_o, z_o).

With the values for the partial derivatives of y and z with respect to x_2 and x_3 at $x_2 = x_3 = 0$, as given in expressions (39) and (40), we can now simplify equations (29), (30) and (31) and obtain the derivatives of x_2 and x_3 with respect to y and z at (y_o, z_o). Taking account

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of only those derivatives of y and z with respect to x_2 and x_3 which do not vanish at $x_2 = x_3 = 0$, equations (29), (30) and (31) become (dropping the subscript zero with it understood from here on that all derivatives stand for their values at (y_o, z_o)):

$$\begin{cases} 1 = y_2 x_{2y} \\ 0 = z_3 x_{3y} \\ 1 = z_3 x_{3z} \\ 0 = y_2 x_{2z} \quad ; \end{cases} \tag{41}$$

$$\begin{cases} 0 = y_2 x_{2yy} \\ 0 = z_{22} x_{2y}^2 + z_3 x_{3yy} \\ 0 = y_2 x_{2zz} \\ 0 = z_{33} x_{3z}^2 + z_3 x_{3zz} \\ 0 = y_{23} x_{2y} x_{3z} + y_2 x_{2yz} \\ 0 = z_3 x_{3yz} \quad ; \end{cases} \tag{42}$$

$$\begin{cases} 0 = y_{222} x_{2y}^3 + 3 y_{23} x_{3yy} x_{2y} + y_2 x_{2yyy} \\ 0 = z_3 x_{3yyy} \\ 0 = y_2 x_{2zzz} \\ 0 = z_{333} x_{3z}^3 + 3 z_{33} x_{3zz} x_{3z} + z_3 x_{3zzz} \\ 0 = y_2 x_{2yyz} \\ 0 = z_{223} x_{2y}^2 x_{3z} + 2 z_{22} x_{2y} x_{2yz} + z_{33} x_{3yy} x_{3z} + z_3 x_{3yyz} \\ 0 = y_{233} x_{3z}^2 x_{2y} + 2 y_{23} x_{2yz} x_{3z} + y_{23} x_{2y} x_{3zz} + y_2 x_{2yzz} \\ 0 = z_3 x_{3zzy} \quad . \end{cases} \tag{43}$$

Solving (41), (42) and (43) in terms of the derivatives of y and z with respect to x_2 and x_3 evaluated at (y_o, z_o) we have

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$$\left\{\begin{aligned} x_{2y} &= \frac{1}{y_2} \\ x_{2z} &= x_{3y} = 0 \\ x_{3z} &= \frac{1}{z_3} \; ; \end{aligned}\right. \tag{44}$$

$$\left\{\begin{aligned} x_{2yy} &= x_{2zz} = x_{3yz} = 0 \\ x_{3zz} &= -\frac{z_{33}}{z_3^3} \\ x_{2yz} &= -\frac{y_{23}}{y_2^2 z_3} \\ x_{3yy} &= -\frac{z_{22}}{z_3 y_2^2} \; ; \end{aligned}\right. \tag{45}$$

$$\left\{\begin{aligned} x_{2yyy} &= -\frac{y_{222}}{y_2^4} - 3\frac{y_{22} z_{22}}{z_3 y_2^4} \\ x_{2yzz} &= -\frac{y_{233}}{y_2^2 z_3^2} + 2\frac{y_{23}^2}{z_3^2 y_2^3} + \frac{y_{22} z_{33}}{z_3^3 y_2^2} \\ x_{2zzz} &= x_{2yyz} = x_{3yzz} = x_{3yyy} = 0 \\ x_{3zzz} &= -\frac{z_{333}}{z_3^4} + 3\frac{z_{33}^2}{z_3} \\ x_{3yyz} &= -\frac{z_{223}}{y_2^2 z_3^2} + 2\frac{z_{23} y_{23}}{y_2^3 z_3^2} + \frac{z_{22} z_{33}}{z_3^3 y_2^2} \; . \end{aligned}\right. \tag{46}$$

If we put the values from (39) and (40) into these equations we then obtain expressions for the derivatives above, which, although unnecessary to the actual calculation, are useful as a means of

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checking and facilitating the calculation of the partials of Λ with respect to y and z. Thus we have for the non-zero partial derivatives of the x^i with respect to the y^α

$$\frac{\partial^n x^2}{\partial y^\alpha \partial y^\beta \cdots} \begin{cases} x_{2y} = \dfrac{a+e}{a} \\[2ex] x_{2yz} = \dfrac{d(a-e)(a+e)^2}{2a^3(d^2-a^2-ae)} \\[2ex] x_{2yyy} = -\dfrac{9e(a+e)^2}{2a^3} + \dfrac{3d^2(2a-e)(a-e)(a+e)^2}{4a^6(d^2-a^2-ae)} \\[2ex] x_{2yzz} = -\dfrac{3e(a+e)^4}{2a^3(d^2-a^2-ae)^2} - \dfrac{d^2(a-e)(a+e)^4}{2a^3(d^2-a^2-ae)^2} - \\ \qquad \dfrac{d^2(a-e)(4a-3e)(a+e)^4}{4a^4(d^2-a^2-ae)^3} + \dfrac{d^4(a-e)(a+e)^3}{a^4(d^2-a^2-ae)^3} \quad ; \end{cases} \tag{47}$$

$$\frac{\partial^n x^3}{\partial y^\alpha \partial y^\beta \cdots} \begin{cases} x_{3z} = -\dfrac{(a+e)^2}{(d^2-a^2-ae)} \\[2ex] x_{3yy} = \dfrac{d(2a-e)(a+e)^2}{2a^3(d^2-a^2-ae)} \\[2ex] x_{3zz} = \dfrac{d(4a-3e)(a+e)^4}{2a(d^2-a^2-ae)^3} - \dfrac{2d^3(a+e)^3}{a(d^2-a^2-ae)^3} \\[2ex] x_{3yyz} = -\dfrac{3e(a+e)^4}{2a^3(d^2-a^2-ae)^2} - \dfrac{d^2(2a-e)(a+e)^4}{2a^3(d^2-a^2-ae)^2} - \\ \qquad \dfrac{d^2(2a-e)(4a-3e)(a+e)^4}{4a^4(d^2-a^2-ae)^3} + \dfrac{d^4(2a-e)(a+e)^3}{a^4(d^2-a^2-ae)^3} \\[2ex] x_{3zzz} = -\dfrac{9e(a+e)^6}{2a(d^2-a^2-ae)^4} - \dfrac{3d^2(3a-2e)(a+e)^5}{a^2(d^2-a^2-ae)^4} + \\ \qquad \dfrac{6d^4(a+e)^4}{a^2(d^2-a^2-ae)^4} - \dfrac{3d^2(4a-3e)^2(a+e)^6}{4a^2(d^2-a^2-ae)^5} + \\ \qquad \dfrac{6d^4(4a-3e)(a+e)^5}{a^2(d^2-a^2-ae)^5} - \dfrac{12d^6(a+e)^4}{a^2(d^2-a^2-ae)^5} \quad . \end{cases} \tag{48}$$

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6. **The partial derivatives of relative phase function with respect to the paraboloid coordinates.**

The next step in obtaining the expansion of Λ in a Taylor's series about (y_o, z_o) is the calculation of the partial derivatives of Λ with respect to y and z, evaluated at (y_o, z_o). Because of the form of equation (21) this is not directly possible. We must therefore do the calculation via equation (27), which requires the derivatives of Λ with respect to x_2 and x_3. Put equation (21) in the form

$$\Lambda = S^{1/2}\left\{\frac{M}{D}\right\} = S^{1/2}T \qquad (49)$$

where $M = 8a^2(x_2^2+x_3^2+4a^2) - 4ae(x_2^2+x_3^2-4a^2) - 16a^2dx_3$

and again

$$S = \left(\frac{x_2^2+x_3^2-4a^2}{4a} - e\right)^2 + x_2^2 + (x_3-d)^2$$

$$D = (x_2^2+x_3^2+4a^2)^2 - 4ae(x_2^2+x_3^2-4a^2) - 16a^2dx_3 .$$

Then if we write $\Lambda_2 = \dfrac{\partial\Lambda}{\partial x_2}$, $\Lambda_3 = \dfrac{\partial\Lambda}{\partial x_3}$, etc. we have

$$\Lambda_2 = \frac{1}{2}\frac{TS_2}{S^{1/2}} + T_2S^{1/2}$$

$$\Lambda_3 = \frac{1}{2}\frac{TS_3}{S^{1/2}} + T_3S^{1/2}$$

$$\text{or } \Lambda_i = \frac{1}{2}\frac{TS_i}{S^{1/2}} + T_iS^{1/2} \qquad (50)$$

$$\Lambda_{ij} = -\frac{1}{4}\frac{S_iS_jT}{S^{3/2}} + \frac{1}{2}\frac{T_jS_i}{S^{1/2}} + \frac{1}{2}\frac{T_iS_j}{S^{1/2}} + \frac{1}{2}\frac{TS_{ij}}{S^{1/2}} + T_{ij}S^{1/2}$$

$$\Lambda_{ijk} = \frac{3}{8}\frac{S_iS_jS_kT}{S^{5/2}} - \frac{1}{4}\frac{T_kS_iS_j + T_jS_iS_k + T_iS_jS_k}{S^{3/2}} - \frac{1}{4}\frac{TS_iS_{jk} + TS_jS_{ik} + TS_kS_{ij}}{S^{3/2}}$$

$$+ \frac{1}{2}\frac{T_{jk}S_i + T_{ik}S_j + T_{ij}S_k}{S^{1/2}} + \frac{1}{2}\frac{T_iS_{jk} + T_jS_{ik} + T_kS_{ij}}{S^{1/2}} + \frac{1}{2}\frac{TS_{ijk}}{S^{1/2}} + T_{ijk}S^{1/2} ,$$

$$(i, j, k = 2, 3) .$$

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and we need to evaluate the partials of S and T and consequently the partials of M, the numerator of T - the denominator, D, being the same as that of Q and R in equation (32) we can use expressions (37) for it. The value of M and its partial derivatives with respect to x_2 and x_3 at $x_2 = x_3 = 0$ are then

$$\begin{aligned}
M\ &\big|_{0,0} = 16a^3(2a+e) \\
M_2\ &\big|_{0,0} = 16a^2x_2 - 8aex_2\big|_{0,0} = 0 \\
M_3\ &\big|_{0,0} = 16a^2x_3 - 8aex_3 - 16a^2d\big|_{0,0} = -16a^2d \\
M_{22}\ &\big|_{0,0} = 16a^2 - 8ae\big|_{0,0} = 8a(2a-e) \\
M_{23}\ &\big|_{0,0} = 0 \\
M_{33}\ &\big|_{0,0} = 16a^2 - 8ae\big|_{0,0} = 8a(2a-e) \\
M_{222}\ &= M_{223} = M_{332} = M_{333} = 0
\end{aligned} \tag{51}$$

and with these values, expressions (37) and the table of derivatives of a quotient found in Appendix A, we have for the value of T and its partial derivatives at $x_2 = x_3 = 0$

$$\begin{aligned}
T &= \frac{(2a+e)}{(a+e)} \\
T_2 &= 0 \\
T_3 &= \frac{d}{(a+e)^2} \\
T_{22} &= -\frac{(2a-e)}{2a(a+e)^2} \\
T_{23} &= 0 \\
T_{33} &= \frac{2d^2}{a(a+e)^3} - \frac{2a-e}{2a(a+e)^2} \\
T_{222} &= 0 \\
T_{223} &= -d\frac{2a-e}{a^2(a+e)^3} \\
T_{332} &= 0 \\
T_{333} &= \frac{6d^3}{a^2(a+e)^4} - \frac{3d(2a-e)}{a^2(a+e)^3}
\end{aligned} \tag{52}$$

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and the value of S and its partial derivatives with respect to x_2 and x_3 at $x_2 = x_3 = 0$

$$\begin{aligned}
S \quad & \Big|_{0,0} = (a+e)^2 + d^2 \\
S_2 \quad & \Big|_{0,0} = \frac{x_2^3 + x_2 x_3^2 - 4a^2 x_2}{4a^2} - \frac{e x_2}{a} + 2x_2 \Big|_{0,0} = 0 \\
S_3 \quad & \Big|_{0,0} = \frac{x_2^2 x_3 + x_3^3 - 4a^2 x_3}{4a^2} - \frac{e x_3}{a} + 2(x_3 - d) \Big|_{0,0} = -2d \\
S_{22} \quad & \Big|_{0,0} = \frac{3x_2^2 + x_3^2 - 4a^2}{4a^2} - \frac{e}{a} + 2 \Big|_{0,0} = \frac{a-e}{a} \\
S_{23} \quad & \Big|_{0,0} = \frac{x_2 x_3}{2a^2} \Big|_{0,0} = 0 \\
S_{33} \quad & \Big|_{0,0} = \frac{x_2^2 + 3x_3^2 - 4a^2}{4a^2} - \frac{e}{a} + 2 \Big|_{0,0} = \frac{a-e}{a} \qquad (53) \\
S_{222} \quad & \Big|_{0,0} = \frac{6x_2}{4a^2} \Big|_{0,0} = 0 \\
S_{223} \quad & \Big|_{0,0} = \frac{x_3}{2a^2} \Big|_{0,0} = 0 \\
S_{332} \quad & \Big|_{0,0} = \frac{x_2}{2a^2} \Big|_{0,0} = 0 \\
S_{333} \quad & \Big|_{0,0} = \frac{6x_3}{4a^2} \Big|_{0,0} = 0
\end{aligned}$$

Using (50) and (53) and (52), we have for the values of Λ and its partial derivatives with respect to x_2 and x_3 at $x_2 = x_3 = 0$ and letting $S^{1/2}\big|_{0,0} = s$,

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$$\Lambda_0 = s\frac{2a+e}{a+e}$$

$$\Lambda_2 = 0$$

$$\Lambda_3 = \frac{d(d^2-a^2-ae)}{(a+e)^2 s}$$

$$\Lambda_{22} = -\frac{e}{(a+e)s} - \frac{d^2(2a-e)}{2a(a+e)^2 s}$$

$$\Lambda_{23} = 0 \tag{54}$$

$$\Lambda_{33} = -\frac{d^2(2a+e)}{(a+e)s^3} - \frac{d^2(2a-5e)}{2a(a+e)^2 s} - \frac{e}{(a+e)s} + \frac{2d^4}{a(a+e)^3 s}$$

$$\Lambda_{222} = 0$$

$$\Lambda_{223} = \frac{d(2a+e)(a-e)}{2a(a+e)s^3} + \frac{d(3a-2e)}{2a(a+e)^2 s} - \frac{d^3(2a-e)}{a^2(a+e)^3 s} - \frac{d(2a-e)}{a^2(a+e)s}$$

$$\Lambda_{332} = 0$$

$$\Lambda_{333} = -\frac{3d^3(2a+e)}{(a+e)s^3} - \frac{3d^3}{(a+e)^2 s^3} + \frac{3d(2a+e)(a-e)}{2a(a+e)s^3} + \frac{3d(3a-2e)}{2a(a+e)^2 s}$$
$$- \frac{3d(2a-e)}{a^2(a+e)s} - \frac{3d^3(3a-2e)}{a^2(a+e)^3 s} + \frac{6d^5}{a^2(a+e)^4 s}$$

7. The value of the partial derivatives of relative phase function with respect to the focal plane coordinates.

We are now in position to calculate the partial derivatives of Λ with respect to y and z, which are the ones required for the expansion of Λ in the Taylor's series in y and z. Writing out equations (27) without the summation convention and with $y^2 = y$, $y^3 = z$, $x^2 = x_2$ and $x^3 = x_3$ also $x_{2y} = \frac{\partial x_2}{\partial y}$, etc.

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$$
\begin{aligned}
\Lambda_y &= \Lambda_2\, x_{2y} + \Lambda_3\, x_{3y} \\
\Lambda_z &= \Lambda_2\, x_{2z} + \Lambda_3\, x_{3z} \\
\Lambda_{yy} &= \Lambda_{22}\, x_{2y}^2 + 2\Lambda_{23}\, x_{2y}\, x_{3y} + \Lambda_{33}\, x_{3y}^2 + \Lambda_2\, x_{2yy} + \Lambda_3\, x_{3yy} \\
\Lambda_{yz} &= \Lambda_{22}\, x_{2y} x_{2z} + \Lambda_{23}(x_{2y} x_{3z} + x_{3y} x_{2z}) + \Lambda_{33}\, x_{3y} x_{3z} + \Lambda_2\, x_{2yz} + \Lambda_3\, x_{3yz} \\
\Lambda_{zz} &= \Lambda_{22}\, x_{2z}^2 + 2\Lambda_{23}\, x_{2z} x_{3z} + \Lambda_{33}\, x_{3z}^2 + \Lambda_2\, x_{2zz} + \Lambda_3\, x_{3zz} \\
\Lambda_{yyy} &= \Lambda_{222} x_{2y}^3 + 3\Lambda_{223}\, x_{2y}^2 x_{3y} + 3\Lambda_{332}\, x_{2y} x_{3y}^2 + \Lambda_{333}\, x_{3y}^3 + 3\Lambda_{22}\, x_{2yy} x_{2y} \\
&\quad + 3\Lambda_{23}(x_{2yy} x_{3y} + x_{3yy} x_{2y}) + 3\Lambda_{33}\, x_{3yy} x_{3y} + \Lambda_2\, x_{2yyy} + \Lambda_3\, x_{3yyy} \\
\Lambda_{yyz} &= \Lambda_{222} x_{2y}^2 x_{2z} + \Lambda_{223}(2x_{2y} x_{3y} x_{2z} + x_{2y}^2 x_{3z}) + \Lambda_{332}(2x_{3y} x_{3z}\, x_{2y} + x_{3y}^2 x_{2z}) \qquad (55) \\
&\quad + \Lambda_{333}\, x_{3y}^2 x_{3z} + \Lambda_{22}(2x_{2yz} x_{2y} + x_{2yy} x_{2z}) + \Lambda_{23}(2x_{2yz} x_{3y} + 2x_{3yz} x_{2y} + x_{2yy} x_{3z} \\
&\quad + x_{3yy}\, x_{2z}) + \Lambda_{33}(2x_{3yz} x_{3y} + x_{3yy} x_{3z}) + \Lambda_2\, x_{2yyz} + \Lambda_3\, x_{3yyz} \\
\Lambda_{zzy} &= \Lambda_{222} x_{2z}^2 x_{2y} + \Lambda_{223}(2x_{2z} x_{3z} x_{2y} + x_{2z}^2 x_{3y}) + \Lambda_{332}(2x_{3z} x_{3y} x_{2z} + x_{3z}^2 x_{2y}) \\
&\quad + \Lambda_{333}\, x_{3z}^2 x_{3y} + \Lambda_{22}(2x_{2yz} x_{2z} + x_{2zz} x_{2y}) + \Lambda_{23}(2x_{2yz} x_{3z} + 2x_{3yz}\, x_{2z} \\
&\quad + x_{2zz} x_{3y} + x_{3zz} x_{2y}) + \Lambda_{33}(2x_{3yz}\, x_{3z} + x_{3zz} x_{3y}) + \Lambda_2\, x_{2zzy} + \Lambda_3\, x_{3zzy} \\
\Lambda_{zzz} &= \Lambda_{222}\, x_{2z}^3 + 3\Lambda_{223}\, x_{2z}^2 x_{3z} + 3\Lambda_{332} x_{3z}^2 x_{2z} + \Lambda_{333}\, x_{3z}^3 + 3\Lambda_{22}\, x_{2zz} x_{2z} \\
&\quad + 3\Lambda_{23}(x_{2zz} x_{3z} + x_{3zz} x_{2z}) + 3\Lambda_{33} x_{3zz} x_{3z} + \Lambda_2\, x_{2zzz} + \Lambda_3\, x_{3zzz} \quad .
\end{aligned}
$$

Rewriting these equations with only the non-zero terms we have

$$
\begin{aligned}
\Lambda_y &= 0 \\
\Lambda_z &= \Lambda_3\, x_{3z} \\
\Lambda_{yy} &= \Lambda_{22}\, x_{2y}^2 + \Lambda_3\, x_{3yy} \\
\Lambda_{yz} &= 0 \qquad\qquad\qquad\qquad (55') \\
\Lambda_{zz} &= \Lambda_{33}\, x_{3z}^2 + \Lambda_3\, x_{3zz} \\
\Lambda_{yyy} &= 0 \\
\Lambda_{yyz} &= \Lambda_{223}\, x_{2y}^2 x_{3z} + 2\Lambda_{22}\, x_{2yz} x_{2y} + \Lambda_{33}\, x_{3yy} x_{3z} + \Lambda_3\, x_{3yyz} \\
\Lambda_{zzy} &= 0 \\
\Lambda_{zzz} &= \Lambda_{333}\, x_{3z}^3 + 3\Lambda_{33}\, x_{3zz} x_{3z} + \Lambda_3\, x_{3zzz}
\end{aligned}
$$

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Then using the values for the partials of x_2 and x_3 with respect to y and z of equations (44), (45) and (46) we have for the non-zero derivatives above

$$\Lambda_z = \frac{\Lambda_3}{z_3}$$

$$\Lambda_{yy} = \frac{\Lambda_{22}}{y_2^2} - \frac{\Lambda_3 z_{22}}{z_3 y_2^2}$$

$$\Lambda_{zz} = \frac{\Lambda_{33}}{z_3^2} - \frac{\Lambda_3 z_{33}}{z_3^3} \tag{56}$$

$$\Lambda_{yyz} = \frac{\Lambda_{223}}{y_2^2 z_3} - \frac{2\Lambda_{23} y_{23}}{y_2^3 z_3} - \frac{\Lambda_{33} z_{22}}{y_2^2 z_3^2} + \Lambda_3\left(-\frac{z_{223}}{y_2^2 z_3^2} + \frac{2y_{23} z_{22}}{y_2^3 z_3^2} + \frac{z_{22} z_{33}}{y_2^2 z_3^3}\right)$$

$$\Lambda_{zzz} = \frac{\Lambda_{333}}{z_3^3} - \frac{3\Lambda_{33} z_{33}}{z_3^4} + \Lambda_3\left(-\frac{z_{333}}{z_3^4} + \frac{3z_{33}^2}{z_3^5}\right)$$

Using either (55'), (54), (47) and (48) or (56), (54), (39) and (40) we can obtain the values of the partial derivatives of Λ with respect to y and z at (y_0, z_0) as the following expressions, in which $s = \sqrt{(a+e)^2 + d^2}$ and $A = (d^2 - a^2 - ae)$,

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$$\frac{\partial\Lambda}{\partial y} = 0$$

$$\frac{\partial\Lambda}{\partial z} = -\frac{d}{s}$$

$$\frac{\partial^2\Lambda}{\partial y^2} = -\frac{e(a+e)}{a^2 s}$$

$$\frac{\partial^2\Lambda}{\partial y\partial z} = 0 \tag{57}$$

$$\frac{\partial^2\Lambda}{\partial z^2} = -\frac{(a+e)^3}{A^2}\left[\frac{e}{s} + \frac{d^2(2a+e)}{s^3} - \frac{d^2}{as}\right]$$

$$\frac{\partial^3\Lambda}{\partial y^3} = 0$$

$$\frac{\partial^3\Lambda}{\partial y^2\partial z} = \frac{d(2a-e)}{2a^3}\ \frac{(a+e)^3}{A^2}\left[\frac{e}{s} + \frac{d^2(2a+e)}{s^3} - \frac{d^2}{as}\right] + \frac{d(a-e)(a+e)^2}{2a^3 A}\cdot \left[\frac{1}{s} - \frac{(2a+e)(a+e)}{s^3}\right]$$

$$\frac{\partial^3\Lambda}{\partial y\partial z^2} = 0$$

$$\begin{aligned}\frac{\partial^3\Lambda}{\partial z^3} = {} & \frac{3d(a+e)^4}{2aA^4}[(4a-3e)(a+e)-4d^2]\left[\frac{e}{s} + \frac{d^2(2a+e)}{s^3} - \frac{d^2}{as}\right] \\ & + \frac{3d(a-e)(a+e)^4}{2aA^3}\left[\frac{1}{s} - \frac{(2a+e)(a+e)}{s^3}\right] + \frac{e}{s}\left[\frac{3d(a-e)(a+e)^4}{a^2 A^3}\right] \\ & + \frac{3d^3(a+e)^4}{s^2 A^3}\left[\frac{1}{s} + \frac{(2a+e)(a+e)}{s^3}\right].\end{aligned}$$

We are now in a position to perform the expansion of $\Lambda(y,z)$ in a Taylor's series in y and z about $(y_o, z_o) = (0, -\frac{ad}{a+e})$ up to the terms of the third order.

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8. The relative phase function as a power series in terms of the focal plane coordinates.

Writing equation (26) without the summation convention and with $y^2 = y$ and $y^3 = z$ we have

$$
\begin{aligned}
\Lambda(y,z) = \Lambda_0 &+ \left(\frac{\partial\Lambda}{\partial y}\right)_0 (y-y_0) + \left(\frac{\partial\Lambda}{\partial z}\right)_0 (z-z_0) + \left(\frac{\partial^2\Lambda}{\partial y^2}\right)_0 \frac{(y-y_0)^2}{2!} \\
&+ 2\left(\frac{\partial^2\Lambda}{\partial y\,\partial z}\right)_0 \frac{(y-y_0)(z-z_0)}{2!} + \left(\frac{\partial^2\Lambda}{\partial z^2}\right)_0 \frac{(z-z_0)^2}{2!} + \left(\frac{\partial^3\Lambda}{\partial y^3}\right)_0 \frac{(y-y_0)^3}{3!} \qquad (26') \\
&+ 3\left(\frac{\partial^3\Lambda}{\partial y^2\,\partial z}\right)_0 \frac{(y-y_0)^2(z-z_0)}{3!} + 3\left(\frac{\partial^3\Lambda}{\partial y\,\partial z^2}\right)_0 \frac{(y-y_0)(z-z_0)^2}{3!} + \left(\frac{\partial^3\Lambda}{\partial z^3}\right)_0 \frac{(z-z_0)^3}{3!} + \dots
\end{aligned}
$$

and with the values for the partial derivatives as given in equation (57) we have

$$
\begin{aligned}
\Lambda(y,z) = s\frac{(2a+e)}{(a+e)} &- \left(z+\frac{ad}{a+e}\right)\left\{\frac{d}{s}\right\} - y^2\left\{\frac{e(a+e)}{2a^2 s}\right\} \\
&- \left(z+\frac{ad}{a+e}\right)^2 \left\{\frac{(a+e)^3}{2A^2}\left[\frac{e}{s} - \frac{d^2}{as} + \frac{d^2(2a+e)}{s^3}\right]\right\} \\
&+ \frac{1}{2}\,y^2\left(z+\frac{ad}{a+e}\right)\left\{\frac{d(2a-e)(a+e)^3}{2a^3A^2}\left[\frac{e}{s} - \frac{d^2}{as} + \frac{d^2(2a+e)}{s^3}\right]\right. \\
&\left. + \frac{d(a-e)(a+e)^2}{2a^3A}\left[\frac{1}{s} - \frac{(2a+e(a+e)}{s^3}\right]\right\} \qquad (58) \\
&+ \frac{1}{5}\left(z+\frac{ad}{a+e}\right)^3 \frac{3d(a+e)^4}{A^3}\left\{\frac{(4a-3e)(a+e)-4d^2}{2aA}\,\cdot\right. \\
&\left[\frac{e}{s} - \frac{d^2}{as} + \frac{d^2(2a+e)}{s^3}\right] + \frac{(a-e)}{2a}\left[\frac{1}{s} - \frac{(2a+e)(a+e)}{s^3} + \frac{2e}{as}\right] \\
&\left. + \frac{d^2}{s^2}\left[\frac{1}{s} + \frac{(2a+e)(a+e)}{s^3}\right]\right\} + \dots\dots
\end{aligned}
$$

This series is being verified by actual computation. The results will be reported separately.

APPENDIX A

DERIVATIVES OF A QUOTIENT

where $\left(\frac{N}{D}\right)_i = \frac{\partial}{\partial x^i}\left(\frac{N}{D}\right)$

$$\left(\frac{N}{D}\right)_i = \frac{N_i}{D} - \frac{ND_i}{D^2}$$

$$\left(\frac{N}{D}\right)_{ii} = \frac{N_{ii}}{D} - 2\frac{N_i D_i}{D^2} + 2\frac{ND_i^2}{D^3} - \frac{ND_{ii}}{D^2} = \frac{N_{ii}}{D} - \frac{2N_i D_i + ND_{ii}}{D^2} + 2\frac{ND_i^2}{D^3}$$

$$\left(\frac{N}{D}\right)_{ij} = \frac{N_{ij}}{D} - \frac{N_i D_j}{D^2} - \frac{N_j D_i}{D^2} - \frac{ND_{ij}}{D^2} + 2\frac{ND_i D_j}{D^3} = \frac{N_{ij}}{D} - \frac{N_i D_j + N_j D_i + ND_{ij}}{D^2} + 2\frac{ND_i D_j}{D^3}$$

$$\left(\frac{N}{D}\right)_{iii} = \frac{N_{iii}}{D} - 3\frac{N_{ii} D_i}{D^2} - 3\frac{N_i D_{ii}}{D^2} + 6\frac{N_i D_i^2}{D^3} - 6\frac{ND_i^3}{D^4} + 6\frac{ND_i D_{ii}}{D^3} - \frac{ND_{iii}}{D^2}$$

$$= \frac{N_{iii}}{D} - \frac{3N_{ii} D_i + 3N_i D_{ii} + ND_{iii}}{D^2} + 6\frac{ND_i D_{ii} + N_i D_i^2}{D^3} - 6\frac{ND_i^3}{D^4}$$

$$\left(\frac{N}{D}\right)_{iij} = \frac{N_{iij}}{D} - \frac{N_{ii} D_j}{D^2} - 2\frac{N_{ij} D_i}{D^2} - 2\frac{N_i D_{ij}}{D^2} + 4\frac{N_i D_i D_j}{D^3} + 2\frac{N_j D_i^2}{D^3} + 4\frac{ND_i D_{ij}}{D^3}$$

$$- 6\frac{ND_i^2 D_j}{D^4} - \frac{N_j D_{ii}}{D^2} - \frac{ND_{iij}}{D^2} + 2\frac{ND_{ii} D_j}{D^3}$$

$$= \frac{N_{iij}}{D} - \frac{N_{ii} D_j + 2N_i D_{ij} + 2N_{ij} D_i + N_j D_{ii} + ND_{iij}}{D^2} + \frac{4N_i D_i D_j + 2N_j D_i^2 + 2ND_{ii} D_j}{D^3}$$

$$\frac{+4ND_i D_{ij}}{} - 6\frac{ND_i^2 D_j}{D^4}$$

$$\left(\frac{N}{D}\right)_{iiii} = \frac{N_{iiii}}{D} - 4\frac{N_{iii} D_i}{D^2} - 6\frac{N_{ii} D_{ii}}{D^2} + 12\frac{N_{ii} D_i^2}{D^3} - 24\frac{N_i D_i^3}{D^4} + 24\frac{N_i D_i D_{ii}}{D^3} - 4\frac{N_i D_{iii}}{D^2}$$

$$- 36\frac{ND_i^2 D_{ii}}{D^4} + 6\frac{ND_{ii}^2}{D^3} + 8\frac{ND_i D_{iii}}{D^3} + 24\frac{ND_i^4}{D^5} - \frac{ND_{iiii}}{D^2}$$

$$= \frac{N_{iiii}}{D} - \frac{4N_{iii} D_i + 6N_{ii} D_{ii} + 4N_i D_{iii} + ND_{iiii}}{D^2} + \frac{12N_{ii} D_i^2 + 24N_i D_i D_{ii} + 8ND_i D_{iii} + 6ND_{ii}^2}{D^3}$$

$$- \frac{24N_i D_i^3 + 36ND_i^2 D_{ii}}{D^4} + 24\frac{ND_i^4}{D^5}$$

DERIVATIVES OF A QUOTIENT

$$\left(\frac{N}{D}\right)_{,iijj} = \frac{N_{iijj}}{D} - 2\frac{N_{iij}D_j}{D^2} - \frac{N_{ii}D_{jj}}{D^2} + 2\frac{N_{ii}D_j^2}{D^3} - 2\frac{N_{ijj}D_i}{D^2} - 4\frac{N_{ij}D_{ij}}{D^2} + 8\frac{N_{ij}D_iD_j}{D^3} - 2\frac{N_iD_{ijj}}{D^2}$$

$$+8\frac{N_iD_{ij}D_j}{D^3} + 4\frac{N_iD_iD_{jj}}{D^3} - 12\frac{N_iD_iD_j^2}{D^4} + 2\frac{N_{jj}D_i^2}{D^3} + 8\frac{N_jD_iD_{ij}}{D^3} - 12\frac{N_jD_i^2D_j}{D^4}$$

$$+4\frac{ND_{ij}^2}{D^3} + 4\frac{ND_iD_{ijj}}{D^3} - 24\frac{ND_iD_jD_{ij}}{D^4} - 6\frac{ND_i^2D_{jj}}{D^4} + 24\frac{ND_i^2D_j^2}{D^5} - \frac{N_{jj}D_{ii}}{D^2}$$

$$-2\frac{N_jD_{iij}}{D^2} + 4\frac{N_jD_{ii}D_j}{D^3} - \frac{ND_{iijj}}{D^2} + 4\frac{ND_{iij}D_j}{D^3} + 2\frac{ND_{ii}D_{jj}}{D^3} - 6\frac{ND_{ii}D_j^2}{D^4}$$

$$= \frac{N_{iijj}}{D} - \frac{2N_{iij}D_j + 2N_{ijj}D_i + N_{ii}D_{jj} + 4N_{ij}D_{ij} + N_{jj}D_{ii} + 2N_iD_{ijj} + 2N_jD_{iij} + ND_{iijj}}{D^2}$$

$$+\frac{2N_{ii}D_j^2 + 8N_{ij}D_iD_j + 2N_{jj}D_i^2 + 8N_iD_{ij}D_j + 4N_iD_iD_{jj} + 4N_jD_{ii}D_j + 8N_jD_iD_{ij} + 4ND_{iij}D_j + 4ND_{ij}^2 + 2ND_{ii}D_{jj} + 4ND_iD_{ijj}}{D^3}$$

$$-\frac{12N_iD_iD_j^2 + 12N_jD_i^2D_j + 6ND_{ii}D_j^2 + 6ND_{jj}D_i^2 + 24ND_{ij}D_iD_j}{D^4} + 24\frac{ND_i^2D_j^2}{D^5}$$

$$\left(\frac{N}{D}\right)_{,iiij} = \frac{N_{iiij}}{D} - \frac{N_{iii}D_j}{D^2} - 3\frac{N_{iij}D_i}{D^2} - 3\frac{N_{ii}D_{ij}}{D^2} + 6\frac{N_{ii}D_iD_j}{D^3} - 3\frac{N_{ij}D_{ii}}{D^2} - 3\frac{N_iD_{iij}}{D^2}$$

$$+6\frac{N_iD_{ii}D_j}{D^3} + 6\frac{N_{ij}D_i^2}{D^3} + 12\frac{N_iD_iD_{ij}}{D^3} - 18\frac{N_iD_i^2D_j}{D^4} - 6\frac{N_jD_i^3}{D^4} - 18\frac{ND_i^2D_{ij}}{D^4}$$

$$+24\frac{ND_i^3D_j}{D^5} + 6\frac{N_jD_iD_{ii}}{D^3} + 6\frac{ND_{ij}D_{ii}}{D^3} + 6\frac{ND_iD_{iij}}{D^3} - 18\frac{ND_iD_{ii}D_j}{D^4}$$

$$-\frac{N_jD_{iii}}{D^2} - \frac{ND_{iiij}}{D^2} + 2\frac{ND_{iii}D_j}{D^3}$$

$$= \frac{N_{iiij}}{D} - \frac{N_{iii}D_j + 3N_{iij}D_i + 3N_{ii}D_{ij} + 3N_{ij}D_{ii} + 3N_iD_{iij} + N_jD_{iii} + ND_{iiij}}{D^2}$$

$$+\frac{6N_{ii}D_iD_j + 6N_{ij}D_i^2 + 6N_iD_{ii}D_j + 12N_iD_{ij}D_i + 6N_jD_{ii}D_i + 2ND_{iii}D_j + 6ND_{iij}D_i + 6ND_{ii}D_{ij}}{D^3}$$

$$-\frac{18N_iD_i^2D_j + 6N_jD_i^3 + 18ND_{ii}D_iD_j + 18ND_{ij}D_i^2}{D^4} + 24\frac{ND_i^3D_j}{D^5}$$